New Sixth Order Mock Theta Functions

Chan Song Heng

The mock theta functions, possibly the most famous and the deepest topic in Ramanujan's lost notebook, were first introduced by Ramanujan in his last letter to G. H. Hardy

In that letter, Ramanaujan gave a list of 17 functions as examples of mock theta functions, classified according to their respective orders. Unfortunately he did not explain what he meant by the "order".

Ramanujan's lost notebook, which G.E. Andrews found in 1976, was seen to contain identities involving mock theta functions of the sixth order and the tenth order. It is widely accepted that the only other mock theta functions discovered besides those of Ramanujan are third order mock theta functions defined by G.N. Watson and G.E. Andrews.

With the complex variable $|q| < 1$, recall the six primary sixth order mock theta functions defined by Andrews and D. Hickerson, namely,

$$\phi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} \prod_{k=1}^{n}(1-q^{2k-1})}{\prod_{k=1}^{2n}(1+q^k)},$$

$$\psi(q) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n^2} \prod_{k=1}^{n-1}(1-q^{2k-1})}{\prod_{k=1}^{2n-1}(1+q^k)},$$

$$\rho(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} \prod_{k=1}^{n}(1+q^k)}{\prod_{k=1}^{n+1}(1-q^{2k-1})},$$

$$\sigma(q) := \sum_{n=0}^{\infty} \frac{q^{(n+2)(n+1)/2} \prod_{k=1}^{n}(1+q^k)}{\prod_{k=1}^{n+1}(1-q^{2k-1})},$$

$$\lambda(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^n \prod_{k=1}^{n}(1-q^{2k-1})}{\prod_{k=1}^{n}(1-q^k)},$$

and

$$\mu(q) := \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{k=1}^{n}(1-q^{2k-1})}{\prod_{k=1}^{n}(1+q^k)}.$$

In a recent collaboration with B. C. Berndt, we conducted a study on the sixth order mock theta functions. We discovered that if we sum the summands of $\rho(q)$ and $\sigma(q)$ over the negative integers, we obtain $\lambda(q)/2$ and $\mu(q)/2$, respectively. However, an examination of the summands of $\phi(q)$ and $\psi(q)$ over the negative and non-positive integers, respectively, leads to the discovery of two new functions, which we defined as

$$\phi_-(q) := \sum_{n=1}^{\infty} \frac{q^n \prod_{k=1}^{2n-1}(1+q^k)}{\prod_{k=1}^{n}(1-q^{2k-1})},$$

$$\psi_-(q) := \sum_{n=1}^{\infty} \frac{q^n \prod_{k=1}^{2n-2}(1+q^k)}{\prod_{k=1}^{n}(1-q^{2k-1})}.$$

We empirically discovered four identities involving these two new functions ϕ_- and ψ_-, which are similar in spirit to identities for sixth order mock theta functions stated by Ramanujan in his lost notebook. We remark that R. J. McIntosh has independently discovered these two mock theta functions and derived transformation formulas for them. However, there are no theorems in McIntosh's work in common with those proved in our paper.

The last entry on page 4 of Ramanujan's lost notebook is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n \prod_{k=1}^{n}(1-q^{6k-3})}{\prod_{k=1}^{n}(1+q^{6k-3})^2} - 2\psi(q) = \frac{\Phi^2(-q)}{2\Psi(q^3)}, \tag{0.1}$$

where

$$\Phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$$

and

$$\Psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}$$

are the generating functions for the sum of squares and sums of triangular numbers respectively. Since the series on the left-hand side of (0.1) does not converge, we provide a meaningful interpretation of this hitherto unexamined entry and proved it using results on ψ by Andrews and Hickerson.

Dr Chan is an Assistant Professor in the Division of Mathematical Sciences. He received his B.Sc from the National University of Singapore in 2001, and his Ph.D. degree from the University of Illinois at Urbana-Champaign in 2005. He completed a two year post-doctoral study at the National University of Singapore before joining NTU in July 2007.

Selected Publications

1. B. C. Berndt, H. H. Chan, S. H. Chan, and W.-C. Liaw, Cranks and dissections in Ramanujan's lost notebook, J. Combin. Theory Ser. A, 109 (2005), no.1, 91–120.
2. B. C. Berndt and S. H. Chan, Sixth order mock theta functions, Adv. Math., 216 (2007), no.2, 771.786.
3. S. H. Chan, A short proof of Ramanujanfs famous ${}_1\psi_1$ summation formula, J. Approx. Theory, 132 (2005), no.1, 149–153.
4. S. H. Chan, Generalized Lambert series, Proc. London Math. Soc., 91 (2005) no.3, 598–622.
5. S. H. Chan, Congruences for Andrews-Paule's broken k-diamond partition function, Discrete Math., 308 (2008), no.23, 5735–5741.